EQUILIBRIUM OF ARBITRARY STEADY FLOWS AT THE TRANSONIC POINTS

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We investigate the behavior of small perturbations of steady state solutions of an arbitrary partial differential equation in two independent variables x and t near a critical point, i.e. near a point where one of the characteristic velocities becomes zero. We assume that all characteristics of the system are real and distinct except t = const, which may become multiple in the case of a parabolically degenerate system.

Critical points coincide with singular points of a system of equations describing steady-state solutions. Possible types of these points were investigated, and we found that in the vicinity of each of these points integral curves lie on a plane passing through that point.

We show that unsteady processes can be described near critical points by a single first order partial differential equation whose coefficients can be determined from the eigenvalues of the singular point of the steady-state equations. Unsteady processes are investigated with nonlinear terms which materially influence the form and amplitude of perturbations taken into account.

A first order equation describing behavior of perturbations near the critical point was obtained for gasdynamic equations and investigated in [1]. Similar methods were also elaborated in works on asymptotic laws of propagation of weak shock waves in gasdynamics [2 to 4] and magnetohydrodynamics [5 and 6].

Another method based on construction of a Green's function was used successfully by S.V. Iordanskii in his Candidate dissertation entitled "On the stability of some self-similar gasdynamic solutions", defended at the Applied Mathematics Institute, Moscow, 1958. He investigated the stability of self-similar unsteady solutions near a point moving with a velocity, which coincided with the characteristic velocity when the self-similar variable was kept constant.

Results obtained in the present paper can be used in investigating the stability of various gasdynamic flows in channels (flows with friction, heat exchange, chemical reactions etc.), magnetohydrodynamic flows and in plasma problems.

Generalization of this theory to the case of parabolically degenerate systems makes it possible to consider cases in media with dissipative processes such as thermal conductivity or limiting conductivity in magnetohydrodynamics.

We have considered as examples quasi-onedimensional magnetohydrodynamic flows with **E** and **H** given, and onedimensional gas flows with finite conductivity in channels of variable cross-section in the presence of a rectilinear magnetic field transverse to velocities.

Consider the following system of Eqs.

$$A_{ij}(u_k, x)\frac{\partial u_j}{\partial t} + B_{ij}(u_k, x)\frac{\partial u_j}{\partial x} + C_i(u_k, x) = 0 \quad (i, j=1, 2, \ldots, n) \tag{1}$$

Here and in the following repeated subscripts will denote summation.

Characteristic velocities c, i.e. velocities of propagation of weak shock waves, are given by

$$|B_{ij}(u_k, x) - cA_{ij}\ (u_k, x)| = 0 \tag{2}$$

We assume that all roots c^k of this equation are real and distinct over some domain of variables x and u_k. We shall also assume that in this region matrix $A_{ij}(u_k, x)$ is of constant rank $m \leqslant n$. System (1) can be hyperbolic or parabolically degenerate as in systems describing dissipative processes.

We shall select a domain of variables x and u_k, in which one of the characteristic velocities e.g. $c^1(x, u_k)$ becomes zero. The rank of the matrix B_{ij} will, on the surface $c^1(x, u_k) = 0$, be equal to $n-1$, while in the remainder of the region matrix B_{ij} will be nondegenerate. A continuous steady-state solution of (1) with c^1 going through zero may exist only if the quantity $C_i\beta_i$ (where β_i are given by $B_{ij}\beta_i = 0$) becomes zero simultaneously with c^1.

If condition $C_i\beta_i = 0$ is not fulfilled, then all derivatives become infinite on the surface $c^1 = 0$ and change their sign on passing through this surface. Consequently, no solution exists which is continuous and single-valued in x. It is easily seen that points at which the relations

$$c^1 = 0, \quad C_i\beta_i = 0$$

hold simultaneously, are the singular points of equations of steady-state solutions. A set of these points forms a $(n-1)$-dimensional surface in the $(x, u_1, \ldots, u_n)$-space.

Let us consider a steady-state solution $u_{k0}(x)$ of (1), in which c^1 changes its sign at some point. We shall assume that x is measured from this point. Behavior of perturbations $v_k(x, t) = u_k(x, t) - u_{k0}(x)$ of this solution is described by the following system of Eqs.

$$A_{ij}\frac{\partial v_j}{\partial t} + B_{ij}\frac{\partial v_j}{\partial x} + F_i = 0 \qquad \left(F_i = C_i - B_{ij}\frac{\partial u_{j0}}{\partial x}\right) \tag{3}$$

where A_{ij}, B_{ij} and F_i depend on x and v_k, and $F_i = 0$ when $v_k = 0$.

We shall assume v_k sufficiently small and consider solutions of (3) on a certain segment $[-\delta, \delta]$. Let us introduce a new variable $x^1 = x/\delta$. Then (3) will become

$$A_{ij}\frac{\partial v_j}{\partial t} + \frac{1}{\delta}B_{ij}\frac{\partial v_j}{\partial x'} + F_i = 0 \tag{4}$$

We next assume that δ is small and seek a solution in the form of a series

$$v_j = v_j^\circ + \delta v_j^1 + \ldots$$

Expanding the coefficients of (4) into series in v_k and x in the zero approximation, we obtain

$$B_{ij}^\circ \partial v_j^\circ / \partial x' = 0 \tag{5}$$

where B_{ij}° is the value of B_{ij} when $v_k = 0$, $x = 0$. Solution of (5) has the form

$$v_j^\circ = w(x', t)\, b_j^\circ + g_j(t) \tag{6}$$

where b_j° are given by

$$B_{ij}^\circ b_j^\circ = 0,$$

while $w(x', t)$ and $g_j(t)$ are arbitrary functions. Solution (6) makes it possible to replace v_k with new variables $w, w_2, \ldots, w_n$, which are such linear combinations of v_k that in the neighborhood of the critical point only w varies significantly with x, while $w_2, \ldots, w_n$ can be assumed, in this neighborhood, to be the functions of time only. We shall see later that w represents a Riemannian invariant corresponding to characteristic velocity c^1.

Quantities $g_j(t)$ in (6) are linear combinations of $w_2(t), \ldots, w_n(t)$.

This means, in the case of steady-state solutions, that in the neighborhood of each singular point integral curves lie on two-dimensional planes defined by Eqs. $w_2 = \text{const}, \ldots, w_n = \text{const}$.

The condition of consistency of the system of first approximation equations yields the following Expression for $w(x', t)$:

$$B_{ij}^\circ\frac{\partial v_j^1}{\partial x'} = -\left(A_{ij}^\circ\frac{\partial v_j^\circ}{\partial t} + \frac{1}{\delta}\Delta B_{ij}\frac{\partial v_j^\circ}{\partial x'} + F_{iv_k}^\circ v_k^\circ\right) \equiv P_i$$
$$\Delta B_{ij} = B_{ij}(x', v_k) - B_{ij}^\circ \tag{7}$$

where A_{ij}° is the value of A_{ij} when $x = 0$, $v_k = 0$ and F_{ivk}° are partial derivatives in v_k of F_i calculated at $x = 0$ and $v_k = 0$. We should note that when F_i is expanded into a series, terms containing v_k in zero degree are absent.

Since B_{ij}° is degenerate, consistency of (7) demands that $P_i\beta_i = 0$ where β_i° are constant magnitudes satisfying

$$B_{ij}^\circ\beta_i^\circ = 0$$

The latter with (6) taken into account, can be written as

$$A_{ij}{}^{\circ}\beta_i{}^{\circ}b_j{}^{\circ}\frac{\partial w}{\partial t}+\frac{1}{\delta}\Delta B_{ij}\beta_i{}^{\circ}b_j{}^{\circ}\frac{\partial w}{\partial x'}+F_{iv_k}^{\circ}\beta_i{}^{\circ}b_k{}^{\circ}w+ + A_{ij}{}^{\circ}\beta_i{}^{\circ}\frac{\partial g_j}{\partial t}+F_{iv_k}^{\circ}\beta_i{}^{\circ}g_k(t)=0 \tag{8}$$

and it can be shown that

$$\Delta B_{ij}\beta_j{}^{\circ}b_j{}^{\circ}\,/\,A_{ij}{}^{\circ}\beta_i{}^{\circ}b_j{}^{\circ}=c^1$$

holds up to the first order terms. Indeed, c^1 is determined from the condition of existence of β_i satisfying the system of algebraic linear Eqs. $c^1 A_{ij}\beta_i = B_{ij}\beta_i$ where $c^1 = 0$ when $v_k = x = 0$. Addition of arbitrarily small increments to v_k and x and contraction of the obtained system with the vector $b_j{}^0$, yields the required equation

Dividing (8) by $A_{ij}{}^{\circ}\beta_i{}^{\circ}b_j{}^{\circ}$ and expanding c^1 in w, $w_2,\ldots, w_n$ and x, we obtain

$$\frac{\partial w}{\partial t}+[c_x{}^1x+c_w{}^1w+\varphi(t)]\frac{\partial w}{\partial x}=Kw+f(t) \tag{9}$$

Functions $\varphi(t)$ and $f(t)$ are given by

$$\varphi(t)=B_{ijv_k}^{\circ}g_k(t)\,\beta_i{}^{\circ}b_j{}^{\circ}\,/\,A_{ij}{}^{\circ}\beta_i{}^{\circ}b_j{}^{\circ}$$

$$f(t)=-\left(A_{ij}{}^{\circ}\beta_i{}^{\circ}\frac{\partial g_j}{\partial t}+F_{iv_k}^{\circ}\beta_i{}^{\circ}g_k(t)\right)/\,A_{ij}{}^{\circ}\beta_i{}^{\circ}b_j{}^{\circ}$$

and represent linear combinations of $w_2,\ldots, w_n$ and their time derivatives. Coefficients $c_x{}^1$, $c_w{}^1$ and K are constants.

Magnitudes $w_2(t),\ldots, w_n(t)$ in terms of which we expressed $\varphi(t)$ and $f(t)$, can be obtained from a solution of the problem falling outside the small neighborhood of the coordinate origin. In many cases we can assume $\varphi(t)$ and $f(t)$ to be equal to zero. Otherwise they can be made equal to zero by the following change of variables:

$$w_1=w-w^*(t),\qquad \xi=x-x^*(t)$$

where $w^*(t)$ and $x^*(t)$ are particular solutions of

$$\frac{dw^*}{dt}=Kw^*+f(t),\qquad \frac{dx^*}{dt}=c_x{}^1x^*+c_w{}^1w^*+\varphi(t)$$

In the following we shall assume that $\varphi(t)$ and $f(t)$ are equal to zero.

Coefficients $c_x{}^1$, $c_w{}^1$ and K in (9) depend on a choice of steady-state solution $u_{j0}(x)$ investigated for stability. To remove this dependence we shall replace w with a new unknown $c \equiv c^1 = c_x{}^1 x + c_w{}^1 w$. Then (9) will become

$$\partial c\,/\,\partial t+c\partial c\,/\,\partial x=\alpha c+\beta x$$

where α and β are constants ($\alpha = K + c_x{}^1$, $\beta = -Kc_x{}^1$). Solution of (10) can be obtained by integrating the following Eqs. of characteristics:

$$dc\,/\,dt=\alpha c+\beta x,\qquad dx\,/\,dt=c \tag{11}$$

It should be noted that in one-dimensional problems of gas dynamics and magnetohydrodynamics with low magnetic Reynold's numbers, $c = u - a \approx a_0\ (M-1)$ where $M = u/a$ and a_0 is the velocity of sound at the critical point of the unperturbed flow.

Change of variables shown above cannot be performed if $c_w{}^1 = 0$ or if the term $c_w{}^1 w\partial w/\partial x$ in (8) is neglected as nonlinear. To obtain a solution in this case, we must use the initial Eq. (9) together with the corresponding Eqs. of characteristics

$$dw\,/\,dt=Kw;\ dx\,/\,dt=c_x{}^1x \tag{12}$$

Equation (10) describes both stationary and nonstationary solutions near a critical point. In the stationary case system (1) gives a solution $c(x)$ of Eq. (10) in the form $c = c(t)$, $x = x(t)$.

The condition that a characteristic Eq. of (11) defining the eigenvalues

$$\lambda^2-\alpha\lambda-\beta=0 \tag{13}$$

has real roots λ_1 and λ_2, is necessary for continuous solutions of (11), passing through the critical point, to exist. We shall assume for definiteness, that $\lambda_1 > \lambda_2$.

Behavior of integral curves on the plane x, c for various real values of λ_1 and λ_2, is shown on Figs. 1 to 3 where arrows indicate the direction of increasing t.

Characteristic directions corresponding to λ_1 and λ_2 are given at the singular point by Eqs. $c = \lambda_1 x$ and $c = \lambda_2 x$.

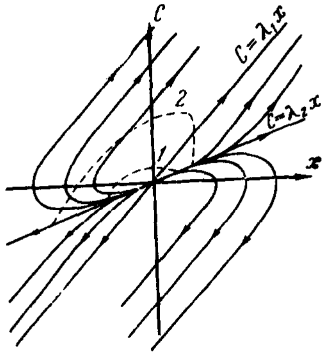

Fig. 1

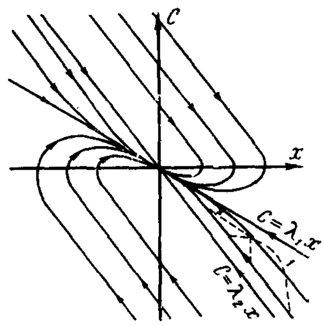

Fig. 2

Any steady-state solution $c_0(x)$ consisting of segments of integral curves singlevalued in x and passing through a singular point, can be taken as an unperturbed solution.

Consider, on the xc-plane, an arbitrary part of an area S bounded by a closed curve whose points move in accordance with (11).

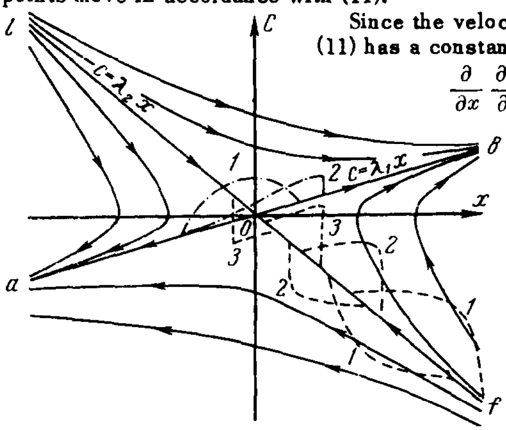

Fig. 3

Since the velocity field given by the right-hand sides of (11) has a constant divergence

$$\frac{\partial}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial}{\partial c}\frac{dc}{dt}=\alpha=\lambda_1+\lambda_2=\frac{1}{S}\frac{dS}{dt}$$

it follows that $S = S_0 e^{\alpha t}$ where S_0 is the surface at $t = 0$.

The area $\int \delta c\, dx$ of an arbitrary perturbation $\delta c = c(x, t) - c_0(x)$ bounded in space where δc is measured from any integral curve of the system (11) selected as an unperturbed solution (Fig. 4), changes in a similar manner. If the perturbation δc is not wholly concentrated on some arbitrarily chosen segment $[x_1, x_2]$, then, when we consider the change of area $\int \delta c\, dx$ on this segment, we must take into account the flux of area through the lines $x = x_1$ and $x = x_2$ so that

$$\frac{d}{dt}\int_{x_1}^{x_2}\delta c\, dx=\alpha\int_{x_1}^{x_2}\delta c\, dx+\int_{c(x_1)}^{c(x_1)+\delta c(x_1)} c\, dc-\int_{c(x_2)}^{c(x_2)+\delta c(x_2)} c\, dc \tag{14}$$

Let us consider an unsteady solution and two closely spaced points whose x-coordinates differ by Δx and values of c by Δc (Fig. 4). Since (11) is a linear system, Δx and Δc also satisfy it. From (11) and Figs. 1 to 3 it follows that, if the initial value of the ratio $\Delta c/\Delta x > \lambda_2$, then it tends with time to a limit value λ_1 which represents a tangent of the angle of inclination of the eigenvector.

Derivative dc/dx tends to a limit as $\exp[-(\lambda_1-\lambda_2)t]$ and the characteristic of an unperturbed solution passing through a singular point tends to the critical point as $\exp\lambda_2 t$ when $\lambda_2 < 0 < \lambda_1$, or as $\exp\lambda_1 t$ when $\lambda_2 < \lambda_1 < 0$.

From this it follows that the approach of a limit slope by the derivative is faster than the approach of the critical point by an unperturbed characteristic when $\lambda_2 < 0 < \lambda_1$ and slower when $\lambda_2 < \lambda_1 < 0$.

If $\Delta c/\Delta x < \lambda_2$, then it becomes infinite in a finite period of time, the continuous solution ceases to exist and a solution with discontinuities must be considered. Thus a perturbation of finite duration in x, tends to assume a saw-tooth or square waveform.

We know [7] that, in the case of weak shock waves, increments of all magnitudes coincide, with accuracy of up to second degree, with increments of magnitudes in a corresponding simple wave. Thus, the presence of a weak shock wave will not lead to the appearance of perturbations of remaining variables $w_2, \ldots, w_n$ behind it.

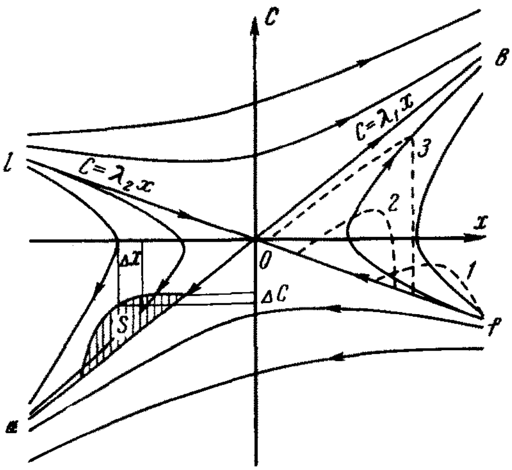

Fig. 4

Velocity of the shock wave is equal, within the same accuracy, to half of the sum of characteristic velocities calculated from the conditions in front and behind the shock wave. From this it follows that the presence of a weak shock wave does not lead to further change in value of the integral $\int \delta c dx$ with time as compared with a continuous case, and Eq. (14) still holds.

Although only hyperbolic systems were investigated in [7], the results obtained are valid for more general systems since the only criterion used in the proof is that the characteristic corresponding to the shock wave is not multiple.

Let us investigate the behavior of solutions of (10) at various combinations of signs of λ_1 and λ_2.

Growth of perturbations $\delta c = c(x, t) - c_0(x)$ with time means that the steady-state solution $u_{k0}(x)$ is unstable; on the other hand, the decay of δc does not imply stability of $u_{k0}(x)$ on a finite segment of the x-axis, since the cause of instability need not be connected with the behavior of the solution near the critical point. Nevertheless we shall assume in the last case, for brevity, that the solution is stable.

1. When $\lambda_1 > \lambda_2 > 0$, we have a singularity in the form of a node with positive characteristic directions. The behavior of integral curves of (11) is given for this singularity, on Fig. 1. From (11) and Fig. 1 we see that any continuous perturbation different from zero at the point $x = 0$ when $t = 0$, grows with time without bounds near this point. The leading and trailing front of such a perturbation will move away from the critical point. If one of these fronts represents a shock wave, then the velocity of its departure from the critical point increases with the increasing intensity of the shock wave.

Thus, any steady-state solution passing through a critical point of the type considered above is unstable. Growth of perturbations leads to establishment of a new steady-state solution in which the sign of c does not change and is the same as the sign of initial perturbation at $x = 0$.

2. When $\lambda_2 < \lambda_1 < 0$, we have a singularity in the form of a node with negative characteristic directions (Fig.2). In the presence of such a singularity, the area and amplitude of any perturbation bounded in space tends to zero, while the leading and trailing fronts move towards the critical point.

If a constant value of the perturbation is maintained on the boundary of the considered region, then, as $t \to \infty$, a new steady-state solution satisfying this boundary condition will be established. If a new boundary value of c, given at the point $x = x_1$ satisfies the inequality $\lambda_2 x_1 < c(x_1) < 0$, then we have a continuous solution passing through the singularity. When $c(x_1) < \lambda_2 x_1$, we have a solution with a shock wave near the coordinate origin.

3. The singularity $\lambda_2 < 0 < \lambda_1$ is a saddle point (Figs. 3 and 4). Four types of steady-state solutions passing through the singularity are possible; they are represented on Fig. 3 by integral curves *aob*, *lof*, *aof* and *lob*.

Consider the perturbation of *aob*. Magnitude δc tends to zero as $\exp \lambda_2 t$. The leading and trailing fronts of the perturbation move away from the critical point. Perturbation of *aob* is shown on Fig. 3 for two consecutive instants of time *1* and *2*, with a dot-dash line.

Perturbations of solution *lof* converge to the critical point. After a sufficiently long time, any perturbation bounded in space assumes a triangular form, one side of which lies on the line *lof*, the second is parallel to *aob* while the third is parallel to the ordinate and represents a shock wave. The side parallel to the line *aob* tends with time to this line. The area of perturbation increases when $\alpha > 0$ and decreases when $\alpha < 0$.

In the last case any perturbations bounded in space tend to zero, and their leading and trailing fronts move towards the critical point. Fig. 3 shows the form of a positive and negative perturbation of the solution *lof* for consecutive instances of time $t_1 < t_2 < t_3 \ldots$ and for $\alpha < 0$ with a broken line.

If $\alpha > 0$, then the development of initial perturbation leads to a rearrangement of the steady-state solution *lof*. Perturbations with positive δc lead to the formation of a shock wave moving to the right and away from the critical point. In this case the solution *lob* is established behind the shock wave. Perturbations with negative δc lead to the solution *aof*. If the initial perturbation contains δc in both signs, then we have solution *aob*. Fig. 4 shows, by means of a broken line, the form of a positive perturbation of the solution *lof* when $\alpha > 0$, for times $t_1 < t_2 < t_3 \ldots$

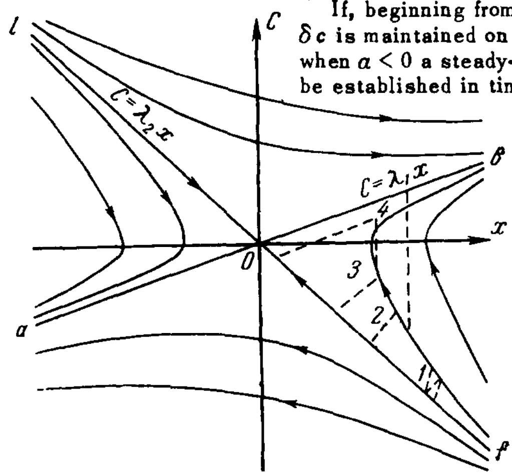

Fig. 5

If, beginning from some instant of time, a constant value of δc is maintained on the boundary of the region in question, then when $\alpha < 0$ a steady-state solution containing a shock wave will be established in time; the distance of the shock wave from the critical point will be directly proportional to the boundary value of δc. Fig. 5 shows the perturbations when δc is positive on the boundary, for consecutive times t_1, t_2, $t_3, \ldots$

Perturbations of the solution *lob* with positive δc decay in the same manner as positive perturbations of the solution *aob*. Perturbations with negative δc grow like the negative perturbations of *lof*. When $\alpha > 0$, this leads to solution *aob*.

Perturbations of solution *aof* with negative δc decay like negative perturbations of solution *aob*, while perturbations with positive δc grow like positive perturbations of solution *lof*; when $\alpha > 0$, this leads to solution *aob*.

Consequently, solution *aob* is always stable in the sense we have considered, while solutions *lof*, *lob* and *aof* are stable when $\alpha < 0$ and unstable when $\alpha > 0$.

Thus stability of a solution near the critical point depends on the character of the singularity of steady-state solutions at this point and is defined by signs of the coefficients α and β of Eq. (13).

When $\alpha < 0$, then any solution is stable near the critical point. When $\alpha > 0$ all steady-state solutions in which $c(x)$ becomes zero are unstable, with the exception of the solution represented by an integral curve passing through a saddle point singularity with dc/dx positive.

It should be noted that if behavior of perturbations in (8) is studied when only terms linear in w are retained, then according to (12) these perturbations will decay when $K < 0$ and grow without bounds when $K > 0$. When nonlinearity is taken into account, perturbations may decay even when $K > 0$, provided that $c_x < 0$ and $\alpha \equiv K + c_x < 0$. The variation in area of a perturbation bounded in x is given in both linear and nonlinear approximation by the same Eq. $S = S_0 \exp \alpha t$.

Different behavior of perturbation amplitudes in the linear and nonlinear approximation is due to the fact that in the nonlinear approximation perturbation tends to assume a triangular form and decrease in the perturbation area is accompanied by the decrease of amplitude. On the other hand, in a linear case the perturbation amplitude may tend to infinity even if its area tends to zero.

Some examples of transonic flows are given below.

Equations connecting the motion and energy of a quasi-one-dimensional flow of electric conducting gas at low values of magnetic Reynold's number (stability of these flows in a linear approximation was studied in [9]) have the form [8]

$$\rho u s = m, \quad \rho u u' + p' + \sigma B (uB - E) = 0$$

$$\left[us\left(\frac{\varkappa}{\varkappa-1}p+\frac{\rho u^2}{2}\right)\right]'+\sigma Es(uB-E)=0 \tag{15}$$

Here s is the cross-section area of the channel, u is the velocity along the x-axis, ρ is density, p is pressure, m is consumption of gas, $\varkappa$ is the ratio of specific heats, E is the electric and B the magnetic field intensity, σ is electric conductivity of the medium and a prime denotes differentiation with respect to x.

Magnitudes s, B and E are given functions of x. Vectors $\mathbf{B}$ and $\mathbf{E}$ are at right angles to each other and lie in a plane orthogonal to the x-axis. $\mathbf{E}$ is normal to the conducting walls and $\mathbf{B}$ is normal to the insulating walls of the channel.

Eqs. (15) were obtained under the assumption that the gas is perfect, nonviscous and not thermal conducting. It follows from [10] that Eqs. (15) can exhibit singularities of all types discussed previously and their appearance depends on the choice of values of the magnitudes defining the flow.

Values of derivatives M'($M = u/a$ is the Mach number) calculated along characteristic directions at the singularity, are found from

$$M'^2-\alpha M'-\beta=0,\quad \alpha=-\frac{\varkappa+1}{2}\frac{\varkappa y\sigma}{mu^2}\left(uB-\frac{\varkappa-1}{\varkappa}E\right)\left(uB-\frac{\varkappa}{\varkappa+1}E\right) \tag{16}$$

Magnitude β is given in terms of B, B', E, E', s, s', s'', $\varkappa$ and σ (see Formula (2.6) of [10] where the conductivity σ was assumed constant). When $\alpha < 0$, flows are stable at the transonic points for any type of singularity.

From (16) it follows that α is negative if one of the following conditions holds:

$$uB>\frac{\varkappa}{\varkappa+1}E,\qquad uB<\frac{\varkappa-1}{\varkappa}E$$

When electrical energy is removed from the system, we always have $\alpha < 0$, since under these conditions the inequality $uB - E > 0$ always holds.

Now we shall consider one-dimensional magnetohydrodynamic flows in a channel of variable cross-section with finite values of magnetic Reynold's numbers. Eqs. describing such flows have the form

$$\rho us=m,\quad \rho uu'+p'+B\theta=0,\quad up'+\varkappa pu'+\varkappa pu\,s'/s-(\varkappa-1)\nu_m\theta^2=0$$
$$u'B+B'u+uBs'/s-\nu_m\theta'=0,\quad B'-\theta=0 \tag{17}$$

where ν_m is magnetic viscosity which in the following will be assumed constant, while remaining symbols are identical with those appearing in (15). In the derivation of (17) magnetic lines of force were assumed to be parallel straight lines normal to the velocity vector.

Behavior of perturbations in the linear approximation was studied for such flows but without sonic transition, in [11] Paper [12] dealt with the formation of shock waves when perturbations were propagated along magnetohydrodynamic flows without the assumption of rectilinearity of magnetic lines of force, but with $\nu_m = 0$ (using adiamatic and stabilisation equations of the magnetic field).

Let us solve Eqs. (17) for derivatives. Replacing one of the variables, say ρ, with the Mach number M, we obtain

$$M'=\frac{M}{2(1-M^2)}\left\{\frac{\varkappa+1}{\varkappa pu}\theta\left[uB+\frac{\varkappa-1}{\varkappa+1}(1+\varkappa M^2)\nu_m\theta\right]-\frac{s}{s'}[2+(\varkappa-1)M^2]\right\}$$
$$u'=\frac{u}{1+\varkappa M^2}\left(2\frac{M'}{M}+\frac{s'}{s}-\frac{B\theta}{p}\right),\quad p'=-(\rho uu'+B\theta)\qquad\left(\rho=\frac{M^2}{\varkappa pu^2}\right) \tag{18}$$
$$\theta'=\frac{1}{\nu_m}\left(u'B+u\theta+\frac{uBs'}{s}\right),\quad B'=\theta$$

Conditions $\theta[uB + (\varkappa - 1)\nu_m\theta] = \varkappa pus'/s$ and $M = 1$ hold at the singularity. By (18) increments in u, p, θ and B can be expressed near the singularity in terms of increments of M and x, while M' calculated along the characteristic directions are given by

$$M'^2-\alpha M'-\beta=0,\qquad \alpha=-\frac{1}{2\varkappa\nu_m pu}(uB+\varkappa\nu_m\theta)[uB+(\varkappa-1)\nu_m\theta]$$
$$\beta=-\frac{\varkappa+1}{4\varkappa\nu_m pu}\left\{\frac{1}{\varkappa+1}\left(\frac{s'}{s}-\frac{B\theta}{p}\right)[uB(b+\varkappa\nu_m\theta)+(\varkappa-1)b\nu_m\theta]+\frac{\nu_m\theta^2}{p}(pu+bB)+\right.$$
$$\left.+u\left(\theta+\frac{s'}{s}B\right)[b+(\varkappa-1)\nu_m\theta]\right\}+\frac{\varkappa+1}{4}\left(\frac{s'}{s}\right)',\qquad b=uB+(\varkappa-1)\nu_m\theta \tag{19}$$

From (19) we see that any type of singularity can be obtained by appropriate choice of the form s of the channel and of θ near this singularity. If the discriminant of (19) is positive, i.e. if $\Delta = \alpha^2/4 + \beta > 0$, then continuous solutions exist, which pass through a singularity.

1) When $\beta > 0$, the singularity is a saddle point; when $\alpha < 0$, both solutions passing through the singularity are stable; when $\alpha > 0$, the solution which has $M' < 0$ (solution *lof* on Fig. 3) is unstable;

2) when $\beta < 0$ and $\alpha < 0$, then the singularity is a node with negative characteristic directions and all solutions passing through it are stable;

3) when $\beta < 0$ and $\alpha > 0$, the singularity is a node with positive characteristic directions and all solutions passing through it are unstable.

We can easily see from (19) that $\alpha < 0$ when one of the following conditions holds:

$$uB < -\varkappa v_m \theta, \qquad uB > -(\varkappa - 1) v_m \theta$$

When $\Delta < 0$, the singularity is a focus and no continuous solutions passing through this singularity exist.

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